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# SL(2,C) Gauge Theory of Gravitation and the Quantization of the Gravitational Field

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A new approach to quantize the gavitational field is presented. It is based on the observation that the quantum character of matter becomes more significant as one gets closer to the big bang. As the metric loses its meaning, it makes sense to consider Schrödinger's three generic types of manifolds—unconnected differentiable, affinely connected, and metrically connected—as a temporal sequence following the big bang. Hence one should quantize the gravitational field on general differentiable manifolds or on affinely connected manifolds. The SL(2,C) gauge theory of gravitation is employed to explore this possibility. Within this framework, the quantization itself may well be canonical.

## 1. INTRODUCTION

Why it is impossible to quantize the gravitational field in the same way that other fields, such as the EM field, are quantized? The answer to this question is well known: The standard quantization procedure applies to fields that are defined over a spacetime of a given metric structure. In the case of the gravitational field, however, there is no metric structure given *a priori*. It is, in fact, this very metric structure that functions as the gravitational field.

The response to this difficulty spans a whole spectrum, from a "minimal" to a "maximal" approach. The "minimal" approach assumes a background of a flat metric structure,

$$g_{\mu\nu} = \eta_{\mu\nu}(\text{flat}) + h_{\mu\nu} \tag{1}$$

and treats the deviations $h_{\mu\nu}$ from flatness as the field to be quantized. In the "maximal" approach one tries for quantization in the context of grand unification.

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## 2. THE PRESENT PROPOSAL

We take the middle ground—it is a way to fully quantize the gravitational field, in all its nonlinearity, but the suggested quantization procedure applies to the gravitational field only.

Consider this: The closer we get to the big bang, the more relevant are the quantum features. On the other hand, as we get down to times of the order of Planck's time, metric time and space lose their meaning.

This last statement is based on the following arguments: In general, matter as we know it does not exist at the Planck time, so we do not have an operational definition of metric relationships.

More specifically, Amati *et al.* (1989) and Konishi *et al.* (1990) found, in the context of string theory, that there is a minimal observable length. A similar result was obtained on more general grounds by N. Itzhaki (private communication). This leads to the following question: Is there a cosmological phase which is generically as well as temporally prior to metric spacetime, and is it possible to quantize gravity at this phase?

Note that if the answer is "yes," we will have a way of resolving the problem that was mentioned at the beginning: The fact that in the case of the gravitational field, unlike the electromagnetic and other fields, we do not have a spacetime of a definite structure on which the field is defined.

The thrust of the present approach is to show that the answer is, indeed, "yes": *There is a cosmological phase which is generically prior to metric spacetime, and it should be possible to quantize gravity at this phase.*

Mathematically, this possibility is based on the SL(2,C) gauge theory of gravitation, as we shall see.

## 3. A CLASSIFICATION OF MANIFOLDS

Consider the following three generic phases leading to metric spacetimes (Schrödinger, 1954):

1. The general differentiable manifold (unconnected)
2. Affinely connected manifold
3. Metrically connected manifold

There is an intermediate phase between manifolds 2 and 3: spacetime with a rudimentary metric structure (STRMS): In metrically connected spacetimes

$$\Gamma^{\alpha}_{\mu\alpha} = (\ln \sqrt{-g})_{,\mu} \equiv \phi_{,\mu} \qquad (2)$$

Hence it is possible, in nonmetric, affinely connected spacetimes, to define $\phi$ as the potential of $\Gamma^{\mu}_{\mu\alpha}$ and let $e^{\phi}$ play the role of $\sqrt{-g}$. Equation (2) determines the volume element up to an overall multiplicative factor. STRMS is metric only in the sense of having a volume element; $ds^2$ is not defined in it.

Perhaps surprisingly, a lot can be done on general differentiable manifolds. Covariant, contravariant, and mixed tensors, tensor densities, and spinors can be defined. (However, the usual relationship between covariant and contravariant components does not hold—there is no metric tensor to connect them!); there are invariant integrals which are obtained by integrating over tensor densities: If £ is a density, then

$$\int £ \, d^4x \tag{3}$$

is invariant, and hence we can have a Lagrangian formalism, with Euler–Lagrange equations.

## 4. THE SL(2,C) GAUGE THEORY OF GRAVITATION

The theory evolved out of the Newman–Penrose null tetrad formalism (Carmeli, 1982, Chapter 3). The latter involves, at each point of spacetime, a tetrad of null vectors $l_\mu$ and $n_\mu$ (real), and $m_\mu$ and $(\bar{m}_\mu$ (complex), which satisfy

$$l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1 \tag{4}$$

the tetrad components of the Weyl and Ricci tensors, and the spin coefficients. For example, the complex tetrad components of the Weyl tensor are

$$\begin{aligned}
\psi_0 &= -C_{\mu\nu\rho\sigma}\, l^\mu m^\nu l^\rho m^\sigma \\
\psi_1 &= -C_{\mu\nu\rho\sigma}\, l^\mu n^\nu l^\rho m^\sigma \\
\psi_2 &= -\frac{1}{2}\, C_{\mu\nu\rho\sigma}\, (l^\mu n^\nu l^\rho n^\sigma - l^\mu n^\nu m^\rho \bar{m}^\sigma) \\
\psi_3 &= -C_{\mu\nu\rho\sigma}\, \bar{m}^\mu n^\nu l^\rho n^\sigma \\
\psi_4 &= -C_{\mu\nu\rho\sigma}\, \bar{m}^\mu n^\nu \bar{m}^\rho n^\sigma
\end{aligned} \tag{5}$$

Carmeli's SL(2,C) gauge theory of gravitation (Carmeli, 1982, Chapter 8) is a group-theoretic formulation of the Newman–Penrose formalism. Both formalisms are equivalent to general relativity, but the SL(2,C) gauge theory of gravitation can be formulated on affinely connected manifolds (Carmeli and Malin, 1977) and, as we found in the context of the present work, the SL(2,C) gauge theory of gravitation can be formulated even on general differentiable manifolds.

## 5. THE STRUCTURE OF THE SL(2,C) GAUGE THEORY OF GRAVITATION

Start out with a *spin frame* defined at each point of a manifold: two linearly independent 2-component spinors satisfying

$$l_A n^A = 1 \tag{6}$$

(Raising and lowering indices are done with the standard antisymmetric spinor matrix.) The components of $l_A$ and $n_A$ are expressed as the components of a $2 \times 2$ matrix $\zeta$. It follows from equation (6) that the matrix $S$ relating any two spin frames,

$$\zeta' = \begin{pmatrix} l'_A \\ n'_A \end{pmatrix} = S^{-1} \begin{pmatrix} l_A \\ n_A \end{pmatrix} = S^{-1}\zeta \tag{7}$$

belongs to the group SL(2,C).

The introduction of local gauge invariance under the transformations $S$ of the spin frame brings about a (compensating) *gauge field* as follows: Since, in general,

$$\nabla_\mu (S^{-1}\zeta) \neq S^{-1}\nabla_\mu\zeta \tag{8}$$

we introduce gauge vector field $2 \times 2$ matrices $B_\mu$ that transform according to

$$B'_\mu = S^{-1} B_\mu S - S^{-1}\partial_\mu S \tag{9}$$

and then

$$(\nabla_\mu - B'_\mu)(S^{-1}\zeta') = S^{-1}(\nabla_\mu - B_\mu)\zeta \tag{10}$$

In analogy with the Yang–Mills field these are called *potentials*.

The corresponding *fields* are defined, also in analogy with Yang–Mills theory, as

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + [B_\mu, B_\nu] \tag{11}$$

where

$$[B_\mu, B_\nu] = B_\mu B_\nu - B_\nu B_\mu \tag{12}$$

The fields transform as follows:

$$F'_{\mu\nu} = S^{-1} F_{\mu\nu} S \tag{13}$$

## 6. THE LAGRANGIAN

There are a number of equivalent ways to write down the Lagrangian density in the context of metric spacetimes. One of these ways can be used in general differentiable manifolds:

$$£ = -\frac{1}{2}\,\mathrm{Tr}\left\{\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}\left(-\frac{1}{2}F_{\gamma\delta} + B_{\gamma,\delta} - B_{\delta,\gamma} + [B_\gamma, B_\delta]\right)\right\} \tag{14}$$

The potentials and fields are considered independent for the purpose of the variational procedure. The Euler–Lagrange equations are

$$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} + [B_\mu, B_\nu] \tag{15}$$

$$\varepsilon^{\alpha\beta\gamma\delta}\{\partial_\delta F_{\alpha\beta} - [B_\delta, F_{\alpha\beta}]\} = 0 \tag{16}$$

In equations (14)–(16) only covariant components of the fields and potentials appear. Hence the absence of correspondence between covariant and contravariant components in nonmetric manifolds is not a problem.

## 7. QUANTIZATION

The straightforward approach is to quantize equation (16) by taking the usual commutation relations between the matrix elements of the $B_\mu$ and their canonical conjugates. It is not clear, however, how to resolve the problem of redundant components, i.e., the existance of more functions than the number of degrees of freedom. There is, however, a different approach, one that worked very well for the linearized equations of general relativity in the Newman–Penrose formalism (Malin, 1974):

The sets of variables of Newman and Penrose are equivalent to the $B_\mu$ and the $F_{\mu\nu}$. In the case of the linearized approximation to the Newman–Penrose formalism, it turns out that all the functions can be expressed in terms of one complex function, $\psi_2$. The problem of quantization reduces, then, to quantizing one complex function. This is done as follows.

Expand in the Wigner matrix elements $D^j_{sm}$ of the irreducible representations of the group SU(2):

$$\psi_2\,(t, r, \theta, \phi) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} \alpha^j_{2m}(t, r)\, D^j_{0m}(\theta, \phi) \tag{17}$$

Using the Newman–Penrose equations, one (eventually) gets a separate partial differential equation for each of the $\alpha^j_{2m}$:

$$\left[3\frac{\partial^2}{\partial t^2} + 2\frac{\partial^2}{\partial t\partial r} - \frac{\partial^2}{\partial r^2} + \frac{j(j+1)}{r^2}\right](r^2\alpha^j_{2m}) = 0 \tag{18}$$

so that the $\alpha^j_{2m}$ can be taken as annihilation operators, and the standard commutation relations between $\alpha^j_{2m}$ and $\overline{\alpha}^j_{2m}$ can be postulated. The generalization of this approach to the nonlinear case is mathematically challenging;

there seems to be no reason of principle, however, that prevents it from being carried out.

## 8. SUMMARY

The following are the key elements of the present approach:

1. Since the quantum characteristics of matter get more significant as one gets closer to the big bang, while the metric characteristics lose their meaning, it makes sense to consider Schrödinger's three generic types of manifolds (unconnected differentiable, affinely connected, and metrically connected) as a temporal sequence following the big bang. (The word "temporal" is used here in the sense of time sequence, not time measurement.)
2. For the same reason it makes sense to try for quantization of the gravitational field on general differentiable manifolds (if possible) or on affinely connected manifolds, with or without rudimentary metric structure.
3. The SL(2,C) gauge theory of gravity seems to be the best existing theory for the exploration of this possibility.
4. Within this framework, the quantization itself may well be canonical.

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